The Abel–Jacobi map in general

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1 Introduction and Preliminaries

In the previous talks we have considered the Abel-Jacobi map $\mathrm{AJ} : \mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}^d_{X/k}$ for projective, smooth, irreducible curves X over algebraically closed fields k. In this talk we will study it in a more general setting. Namely, for a scheme S we will define the Abel-Jacobi map for proper, finitely presented schemes X over S which which are S-flat. Goal of this talk is to understand its scheme-theoretic fibers. In Proposition 9 we will see that they are given by projective spaces.

Unless stated otherwise, $(X, \mathcal{O}_X)$ and $(S, \mathcal{O}_S)$ will be schemes and $f : X \to S$ will be a morphism of schemes in this whole talk.

Definition 1. An $\mathcal{O}_X$-module is *locally of finite presentation* if it is quasi-coherent and locally isomorphic to the cokernel of a homomorphism of type $\mathcal{O}_X^m \to \mathcal{O}_X^n$.

Definition 2. Let $f : X \to S$ proper and of finite presentation and let $\mathcal{F}$ an $\mathcal{O}_X$-module locally of finite presentation which is flat over S.

(i) Then $\mathcal{F}$ is *cohomologically flat over S in dimension 0* if the formation of the direct image $f_*(\mathcal{F})$ commutes with base change along any morphism of schemes $T \to S$.

(ii) We say that f is *cohomologically flat over S in dimension 0* if $\mathcal{O}_X$ is cohomologically flat over S in dimension 0.

Example 1. Let k an algebraically closed field, let X a smooth, projective, irreducible curve over k, let T a scheme over k and let $\mathcal{L} \in \mathrm{Pic}(X_T)$. In the previous talk we saw that the degree map $t \to \mathcal{L}_t$ is locally constant. Now, let $d = \deg \mathcal{L} > 2g - 2$ where g is the genus of the curve. We want to show that in this case, $\mathcal{L}$ is cohomologically flat over $\mathrm{Spec}\, k$ in dimension 0. We assume that $T = \mathrm{Spec}(A)$ is open affine. Let $K^\cdot = (K^0 \to K^1)$ be a complex of finitely generated projective A-modules such that for all A-algebras B,

$$H^p(K^\cdot \otimes_A B) \simeq H^p(X, \mathcal{L}), \quad p \geq 0.$$

We want to show that $H^1(X, \mathcal{L}) = 0$, i.e. $\mathrm{coker}(d^0 \otimes_A B) = 0$. By Nakayama's lemma it suffices to show this claim for $B = k(p)$, $p \in \mathrm{Spec}(A)$. Thus, we may assume that $T = \mathrm{Spec}\, k$. By Serre duality

$$h^1(X, \mathcal{L}) = h^0(X, \mathcal{L}^* \otimes \omega_X)$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ and ω_X is the canonical line bundle. But $\deg(\omega_X) = 2g - 2$ and $\deg(\mathcal{L}^*) < 2 - 2g$. Thus, $h^1(X, \mathcal{L}) = 0$, which shows cohomological flatness in dimension 0.

We will now state some basic result on the direct image of $\mathcal{O}_X$-modules which are locally of finite presentation. This statement is used in the proof of Proposition 9.

Theorem 3 (cf. [BLR90, Thm. 8.1.7])**.** *Let $f : X \to S$ proper and finitely presented. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of locally finite presentation which is S-flat.*

(i) *There existst an $\mathcal{O}_S$-module $\mathcal{Q}$ of locally finite presentation, unique up to canonical isomorphism, such that there exists an isomorphism of functors*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M}) \simeq \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{M})$$

which is fuctorial for all quasi-coherent $\mathcal{O}_S$-modules $\mathcal{M}$.

(ii) *Taking global sections, there exists an isomorphism of functors*

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{M}).$$

(iii) *The $\mathcal{O}_S$-module $\mathcal{Q}$ is locally free if and only if $\mathcal{F}$ is cohomologically flat over S in dimension 0. In this case, $\mathcal{Q}$ and $f_*(\mathcal{F})$ are dual to each other and, in particular $f_*(\mathcal{F})$ is locally free.*

2 Fibers of the Abel-Jacobi map

Definition 4. For a ringed space $(X, \mathcal{O}_X)$ we define the *sheaf of invertible elements* $\mathcal{O}_X^*$ as the presheaf whose local sections are the local sections of $\mathcal{O}_X$ which are invertible.

It is easily seen that $\mathcal{O}_X^*$ is indeed a sheaf. We will now recall some definition and introduce some notation.

Definition 5. (i) A *realtive effective Cartier divisor* on X over S is an effective Cartier divisor D on X such that $D \to X$ is a flat morphism of schemes.

(ii) Let $\mathrm{Div}(X/S)$ the set of all relative effective Cartier divisors on X/S.

Definition 6. Let $f : X \to S$ flat, quasi-compact, quasi-separated and assume that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally, i.e. it holds after any base change. Assume that f admits a section. Define the *relative Picard functor* $\mathrm{Pic}_{X/S}$ via

$$\mathrm{Pic}_{X/S} : (\mathrm{Sch}\,/S)^{\mathrm{op}} \to (\mathrm{Sets}), \quad T \mapsto \mathrm{Pic}(X \times_S T)/\,\mathrm{Pic}(T)$$

The Abel-Jacobi map. Relative effective Cartier divisors are stable under any base change $S' \to S$ by [BLR90, Lemma 8.2.6]. This yields a functor

$$\mathrm{Div}_{X/S} : (\mathrm{Sch}\,/S)^{\mathrm{op}} \to (\mathrm{Sets}), \quad S' \mapsto \mathrm{Div}(X \times_S S'/S').$$

We obtain a canonical morphism

$$\mathrm{AJ} : \mathrm{Div}_{X/S} \to \mathrm{Pic}_{X/S}, \quad D \mapsto \mathcal{O}_X(D),$$

the *Abel-Jacobi map.*

If X is proper, finitely presented and flat over S, $\mathrm{Div}_{X/S}$ is an open subfunctor of the Hilbert functor $\mathrm{Hilb}_{X/S}$ by [BLR90, Lemma 8.2.6]. We saw in talk 4 that if X is a projective, smooth, irreducible curve over an algebraically closed field k, then $\mathrm{Div}_{X/S} = \mathrm{Hilb}_{X/S}$.

Definition 7. Let $\mathcal{C}$ a category admitting fiber products, let $F, G\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$ functors and let $a : F \to G$ a morphism of functors. We say that a is *representable* or F is *relatively representable over* G if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any natural transformation $\xi : hu \to G$ the functor $h_U \times_G F$ is representable, where h_U is the functor $\mathrm{Hom}_{\mathcal{C}}(-, U)$.

Reminder. *Let $n \in \mathbb{N}$, then*

$$\mathbb{P}^n_{\mathbb{Z}}(X) \simeq \{(\mathcal{L}, \alpha) \mid \mathcal{L} \in \mathrm{Pic}(X), \alpha : \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L}\}/\simeq .$$

Projective bundles. For a finitely generated quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$ consider the functor

$$F_{\mathcal{F}} : (\mathrm{Sch}\,/S)^{\mathrm{op}} \to (\mathrm{Sets}), \quad T \mapsto \{(\mathcal{L}, \alpha) \mid \mathcal{L} \in \mathrm{Pic}(T), \alpha : f^*\mathcal{F} \twoheadrightarrow \mathcal{L}\}/\simeq .$$

This functor is representable by a scheme, the so-called *projective bundle* $\mathbb{P}(\mathcal{F})$.

Definition 8. The *geometric fiber* of a morphism of schemes $f : X \to S$ at $s \in S$ is defined to be $X_{\bar{s}} = F \times_S \overline{k(s)}$.

Proposition 9 (cf. [BLR90, Prop. 8.2.7])**.** *Let $f : X \to S$ proper and of finite presentation and let S quasi-compact. Assume that f is flat and that its geometric fibers are reduced and irreducible. Let $\mathcal{L}$ be a line bunlde on $X_T = X \times_S T$, and let $T \to \mathrm{Pic}(X/S)$ the morphism corresponding to $\mathcal{L}$. Then there exists an $\mathcal{O}_T$-module which is locally of finite presentation such that $\mathrm{Div}_{X/S} \times_{\mathrm{Pic}_{X/S}} T$ is represented by the projective T-scheme $\mathbb{P}(\mathcal{F})$.*
Furthermore, there is a canonical way to choose $\mathcal{F}$. If $\mathcal{L}$ is cohomologically flat in dimensio 0, then $f_*(\mathcal{L})$ and $\mathcal{F}$ are locally free and dual to each other.

Example 2. Let k an algebraically closed field, let $f : X \to \mathrm{Spec}\, k$ a projective, smooth, irreducible curve and let $\mathcal{L}$ a line bundle on $X = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k$ of degree $> 2g - 2$, where g is the genus of the curve.
By Example 1, $\mathcal{L}$ is cohomologically flat in dimension 0. Thus, the fiber of the Abel-Jacobi map $\mathrm{Hilb}_{X/S} \times_{\mathrm{Pic}_{X/S}} \mathrm{Spec}\, k$ is given by $\mathbb{P}(\mathcal{F})$ by Proposition 9, where $\mathcal{F}$ is the dual of $f_*\mathcal{L}$. As $f_*\mathcal{L}(\mathrm{Spec}\, k) = H^0(X, \mathcal{L})$ is a finite dimensional vector space, $\mathcal{F}(\mathrm{Spec}\, k) = H^0(X, \mathcal{L})^*$ is a finite dimensional vector space. If n denotes its dimesnion, the fiber is given by $\mathbb{P}(\mathcal{F}) = \mathbb{P}^{n-1}_k$.

References

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